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GAME PROBLEM OF THE "SOFT" IMPULSE CONTACT OF TWO MATERIAL POINTS

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We consider the game problem [1 - 4] of the "soft" (with respect to coordinates and velocities) contact of two material points subjected to the action of given forces of attraction to a fixed center and to the action of control forces. The given forces are proportional to the distances of the points from the center or are equal to zero, while the control forces are arbitrary in direction and bounded in momentum. The first and the second points are treated as the minimizing and the maximizing players in a game where the payoff is the time upto the soft contact. In the process of solving the problem the whole space of game positions is divided into two regions. In the first region we find the solution of the minimax problem and we synthesize three functions of the position: the optimal controls of the first and second players and the time upto soft contact. In the second region we synthesize the second player's control allowing him to avoid the soft contact under any action of the first player.

1. Let the equations of motion of the system have the form

$$x^{\cdot} = y, \qquad y^{\cdot} = -k^2 x + u + v \tag{1.1}$$
$$\mu^{\cdot} = -|u|, \quad \nu^{\cdot} = -|v|, \quad \mu \geqslant 0, \quad \nu \geqslant 0$$

where x, y, u, v are three-dimensional vectors, $|x|$, $|y|$, $|u|$, $|v|$ are their Euclidean moduli, while μ and ν are nonnegative numbers subject to the phase constraints $\mu \geqslant 0, \nu \geqslant 0$ Equations (1.1) can be interpreted as the equations of the relative motion of two material points with masses m_1 and m_2, whose radius-vectors relative to a fixed center O equal r_1 and r_2, while the points at which the given forces $F_1 = -m_1 k^2 r_1$, $F_2 = -m_2 k^2 r_2$ and the control forces $f_1 = m_1 u$, $f_2 = -m_2 v$ act are restricted by the impulse constraints

$$\mu^\circ - \int_0^\tau |u|\, dt = \mu(\tau) \geqslant 0 \tag{1.2}$$
$$\nu^\circ - \int_0^\tau |v|\, dt = \nu(\tau) \geqslant 0$$

In Eqs. (1.1), $x = r_1 - r_2$, $y = r_1 - r_2$, and the constant $k \geqslant 0$ are assumed nonnegative.

We introduce into consideration the three vectors

$$w(\tau) = [x_i(\tau),\ y_i(\tau - 0),\ \mu(\tau - 0),\ \nu(\tau - 0)]$$
$$w^{(2)}(\tau) = [x_i(\tau),\ y_i(\tau - 0) + \nu_{1i},\ \mu(\tau - 0),\ \nu(\tau - 0) - |\nu_1|] =$$
$$= [x_i^{(2)}(\tau),\ y_i^{(2)}(\tau),\ \mu^{(2)}(\tau),\ \nu^{(2)}(\tau)]$$

$$w^{(2,1)}(\tau) = [x_i^{(2)}(\tau),\ y_i^{(2)}(\tau) + \mu_{1i},\ \mu^{(2)}(\tau) - |\mu_1|,\ \nu^{(2)}(\tau)] = \qquad (1.3)$$
$$= [x_i^{(2,1)}(\tau),\ y_i^{(2,1)}(\tau),\ \mu^{(2,1)}(\tau),\ \nu^{(2,1)}(\tau)]$$

These vectors have the following meaning. The first of them, $w(\tau)$ is the left limit as $t \to \tau - 0$ of a certain solution of system (1.1) under finite u, v and such that for $t < \tau$ the solution is continuous and differentiable for almost all t on the half-interval $t \in [\tau - \varepsilon_1,\ \tau)$, $\varepsilon_1 > 0$ The vector $w(\tau)$ is called a game position at $t = \tau$, and in order that the initial value $w(0)$ be a position we ascribe to it a small ($t \in [-\varepsilon,\ 0)$) prehistory with $u(t) = v(t) = 0$ The vector $w^{(2)}(\tau)$ represents the result of the second point's (second player's) impulse actions $v(\tau) = \nu_1\delta$, while the vector $w^{(2,1)}(\tau)$ represents the image of the vector $w^{(2)}(\tau)$, obtained as the result of the first point's (first player's) impulse control $u(w^{(2)}(\tau)) = \mu_1\delta$. The index $i = 1, 2, 3$ indicates the projection of the vectors x, y, μ_1, ν_1 onto the fixed axes. We shall omit the argument τ when this does not give difficulty in comprehension.

Let the pair of controls $[u(w^{(2)}),\ u(w, v)]$, $v(w)$ be such that for $t \geqslant 0$ it generates a trajectory $w(t > 0,\ \{[u(w^{(2)}),\ u(w, v)],\ v(w)\},\ w(0))$ which for almost all t satisfies system (1.1) together with the phase estimates $\nu \geqslant 0$, $\mu \geqslant 0$, is everywhere right continuous in t and has a finite number of jumps in accordance with formulas (1.3). Such a control pair and trajectory are said to be admissible and we shall solve the problem only in this class. The notation $[u(w^{(2)}),\ u(w, v)]$ denotes that the first player can form either the control $u(w^{(2)})$ or, for a finite $v(w)$, the control $u(w, v)$, i.e. the second player is discriminated against. Let us state the problem of soft ($x = y = 0$) contact.

D e f i n i t i o n 1. If the vector $w^{(2)}(\tau) \in M_0\ [|x| = 0]$ and if there exists an impulse control $u(w^{(2)}) = \mu_1(w^{(2)})\ \delta$ which makes the vector $y^{(2,1)}(\tau)$ vanish, then we say that $w^{(2)}(\tau)$ and τ are the vector and the instant of contact respectively. We can verify that when the relations

$$|x(\tau)| = 0, \qquad \mu(\tau) - \nu(\tau) - |y(\tau)| \geqslant 0$$

are fulfilled, any admissible ($\nu(\tau) - |\nu_1| \geqslant 0$) vector $w^{(2)}(\tau)$ corresponds to a contact. On the other hand, when the relations

$$|x(\tau)| = 0, \qquad \mu(\tau) - \nu(\tau) - |y(\tau)| < 0$$

are fulfilled, the vectors

$$w^{(2)}(|y| \neq 0) = [x^{(2)}(\tau),\ y_i^{(2)}(\tau) = y_i(\tau) + \nu y_i/|y|,\ \mu^{(2)}(\tau),\ \nu^{(2)}(\tau) = 0]$$
$$w^{(2)}(|y| = 0) = [x^{(2)}(\tau),\ |y^{(2)}(\tau)| = \nu;\ y_{3,1}^{(2)} = 0,\ \mu^{(2)}(\tau),\ \nu^{(2)}(\tau) = 0]$$

are not vectors of contact. This means that the second player cannot avoid choosing a vector of contact (cannot avoid contact) if and only if the position

$$w(\tau) \in M\ [|x| = 0;\ \mu - \nu - |y| \geqslant 0]$$

In this connection we name M the game termination set.

We pose two fundamental conflict problems.

P r o b l e m 1. Find the controls $u^\circ(w, v)$, $v^\circ(w)$ for which the time $T[u, v]$ of the first hitting of the trajectory onto M satisfies the estimates

$$T[u^\circ, v] \leqslant T[u^\circ, v^\circ] \leqslant T[u, v^\circ]$$

The collection of positions for which a solution of Problem 1 exists forms a set $W^\circ(w)$.

P r o b l e m 2. Find $v_0(w)$ such that the trajectory $w(t,\ \{u,\ v_0(w)\},\ w(0))$ does

not hit onto M for $t \in [0, \infty]$ and for any control u.
The conditions for the existence of $v_0(w)$ delineate a set $W_0(w)$.

2. By ξ we denote the difference $\mu - \nu$, while by j_α, j_β, j_γ we denote the right unit triple of basis vectors. For $|x| \neq 0$ the basis vector j_α is parallel to the vector x, the basis vector j_β is directed along the vector y^β – the projection of vector y onto a plane orthogonal to x. The basis vector j_γ completes the triple. If $y^\beta = 0$, then j_β, j_γ are arbitrary. If $|x| = 0$, then the whole triple is arbitrary. The projections of any vector e onto the basis vectors of the triple are denoted e_α, e_β, e_γ. The results of [1, 4] suggest forming the impulse control

$$u_{(1)}(w^{(2)}) = \lambda_1(w^{(2)})\,\delta j_\alpha - y_\beta^{(2)}\delta j_\beta = \mu_{(1)}\delta \tag{2.1}$$

Here $\lambda_1(w^{(2)})$ is the smallest root of the equation

$$\varphi(w^{(2)}, \lambda) = \xi - \sqrt{(y_\beta^{(2)})^2 + \lambda^2} - \sqrt{(y_\alpha^{(2)} + \lambda)^2 + k^2|x|^2} = 0 \tag{2.2}$$

on the segment $c\,[\lambda^2 \leqslant \xi^2 - y_\beta^2]$. When $k > 0$, we can obtain $k = 1$ by changing the scales of the variables. Therefore, in what follows we shall consider only the two distinct cases: $k = 1$, $k = 0$. To abbreviate the notation we assume that $w^{(2)} = w$. As we shall see subsequently (Lemma 3.2), the assumption that $v(w) = \nu_1\delta = 0$ $(w^{(2)} = w)$ does not affect the possibility of forming a control $u_{(1)}(w^{(2)})$ for $w^{(2)} \neq w$.

Let us divide the whole position space W into the following regions. for the cases $k = 1$, $k = 0$:

$$D^{(1)}\,[R(w,\ k=1) = R^{(1)} = \xi - \sqrt{y^2 + x^2 + 2|x|y_\beta} \geqslant 0;\ |x| > 0]$$
$$y^2 = |y|^2, \quad x^2 = |x|^2$$
$$D^{(0)} = D^{(0,1)}\,[R(w,\ k=0) = R^{(0)} = \xi - |y| > 0;\ |x| > 0] \cup$$
$$\cup\, D^{(0,2)}\,[R^{(0)} = y_\beta = 0;\ y_\alpha < 0;\ |x| > 0]$$
$$D_{(1)} = W \setminus [D^{(1)} \cup M] = D_{(1)}\,[R^{(1)}(w) < 0]$$
$$D_{(0)} = W \setminus [D^{(0)} \cup M] = D_{(0,1)}\,[R^{(0)} < 0] \cup$$
$$\cup\, D_{(0,2)}\,[R^{(0)} = 0;\ |y| > 0;\ y_\beta\,|x| / |y| > 0] \cup$$
$$\cup\, D_{(0,3)}\,[R^{(0)} = y_\beta = 0;\ y_\alpha \geqslant 0;\ |x| > 0]$$

Lemma 2.1 contains four assertions.

2.1.1. When $w \in D^{(1),(0)}$, Eq. (2.2) has the roots $\lambda_1(w,\ k=1) = \lambda^{(1)}(w)$, $\lambda_1(w,\ k=0) = \lambda^{(0)}(w)$ on the segment $C\,[\lambda^2 \leqslant \xi^2 - y_\beta^2]$.

2.1.1. These roots are given by the formulas

$$\lambda^{(1)}(w) = \left(y_\alpha s^{(1)} - \sqrt{\xi^2((s^{(1)})^2 + y_\alpha^2 x^2 - x^2\xi^2)}\right) q^{-1} - y_\alpha$$
$$2s^{(1)} = \xi^2 + x^2 - y^2, \qquad q = \xi^2 - y_\alpha^2 \tag{2.3}$$
$$\lambda^{(0)}(w) = -{}^1/_2\left((\xi + y_\alpha)^2 - y_\beta^2\right)(\xi + y_\alpha)^{-1}, \quad w \in D^{(0,1)}$$
$$\lambda^{(0)}(w) = 0, \qquad w \in D^{(0,2)}$$

2.1.3. The estimates

$$\lambda^{(1)}(w) \leqslant -y_\alpha y_\beta(|x| + y_\beta)^{-1} = \lambda_2(w), \qquad \lambda^{(0)}(w) \leqslant 0 \tag{2.4}$$

are valid.

2.1.4. If the controls $v^{(1),(0)}(w)=0$, then the controls $u_{(1)}(w)$ take the trajectory onto set M in the times

$$T^{(1)}(w)=T[u_{(1)}(w),0]=\pi+\arccos(p^{(1)}(p^{(1)})^2+x^2)^{-1/2}) \qquad (2.5)$$

$$T^{(0)}(w)=T[u_1(w),0]=-|x|/p^{(0)}$$

$$p^{(1)}(w)=y_\alpha+\lambda^{(1)}(w) \qquad p^{(0)}_{(w)}=y_\alpha+\lambda^{(0)}(w)$$

Before we start on the proof let us make the terminology more precise. The equalities

$$w^{(2.1)}(0)=[x(0),\ y_\alpha^{(2.1)}=p^{(1),(0)}(w(0)),\ y_\beta^{(2.1)}=0,\ \xi^{(2.1)}=\xi-|\mu_{(1)}|] \qquad (2.6)$$

are valid on the pair $u_{(1)}(w,v)$, $v(w)=0$. From what follows we shall see that when $t>0$ the control $u_{(1)}(w)$ vanishes and the motion takes place as if the initial position were $w'(0)=w^{(2.1)}(0)$. It is necessary to introduce such a stipulation because $w^{(2.1)}(0)$ is not a position (*). We prove Lemma 2.1 in the order of the assertions made.

2.1.1. Proof. Simple calculations permit us to establish the equalities

$$R^{(1),(0)}(w)=\max_\lambda\varphi(w,k=1,0)),\quad \lambda\in(-\infty,+\infty) \qquad (2.7)$$

$$R^{(1)}(w)=\varphi(w,\lambda_2(\omega)=-y_\alpha y_\beta(|x|+y_\beta)^{-1}) \qquad (2.8)$$

$$R^{(0)}(w)=\varphi(w,-y_\alpha) \qquad (2.9)$$

We can also verify that $\lambda=\lambda_2(w)$ is the unique point of maximum, as is $\lambda=-y_\alpha$ when $y_\beta>0$. If $y_\beta=0$, then the maximum of $R^{(0)}(w)$ is achieved on the segment $y_\alpha+\lambda\leqslant$ $\leqslant 0$. In the regions $D^{(1),(0)}$ we can establish also the estimates

$$\xi^2-y_\beta^{(2)}>0,\qquad \varphi(w,\pm\sqrt{\xi^2-y_\beta^2})\leqslant 0 \qquad (2.10)$$

$$\lambda_2^2(\omega)\leqslant\xi^2-y_\beta^2$$

Equality (2.7) guarantees the existence of a root of Eq. (2.2) on the straight line $\lambda\in(-\infty,+\infty)$. Equalities (2.8), (2.9) together with estimates (2.10), show that when $R^{(1),(0)}>0$ the segment "C" contains not less than two roots, while when $R^{(1),(0)}=0$ not less than one root.

2.1.1. Proof. The replacement $p=\lambda+y_\alpha$ and simple manipulations allow us to obtain corollaries of Eq. (2.2) for $k=1,\ 0$ in the form

$$(\xi^2-y_\alpha^2)p^2-2s(1)y_\alpha p+x^2-(s(1))^2=0 \qquad (2.11)$$

$$(\xi^2-y_\alpha^2)p^2-(\xi^2-y^2)y_\alpha p-{}^1/_2(\xi^2-y^2)=0 \qquad (2.12)$$

When $R^{(1),(0)}>0$ Eqs. (2.11) and (2.12) have precisely two roots each and the smallest ones of each pair are given by formulas (2.3). The arguments used in the proof of assertion 2.1.1 convince us that these pairs of roots satisfy Eq. (2.2) when $k=1{,}0$. If $R^{(1)}=$ $=0$, then (2.2) and (2.11) have the unique root $\lambda^{(1)}(w)=\lambda_2(w)$. If $R^{(0)}=y_3=0$, then (2.12) turns into an identity, however, Eq. (2.2) takes the form

$$\xi-|\lambda|-|y_\alpha+\lambda|=0$$

and its smallest root $\lambda^{(0)}(w)=0$.

2.1.3. Proof. The estimates in 2.1.3. are simple corollaries of the preceding arguments.

*) In what follows "the motion starting from the position $w^{(2.1)}(w,u(w,v),v(w))$" is to be understood in the sense that it takes place just as if it had started from the position $w'(0)=w^{(2.1)}(0)$. This refinement extends to the whole article.

2.1.4. Proof. After realization of control $u_{(1)}$ (w) the motion starts from the position $w^{(2.1)}$ (0) given by formulas (2.6) and, for $t>0$ takes place in accordance with the equations

$$x/|x| = x(0)/|x(0)|,\quad |x|^{\cdot} = y_{\alpha},\quad y_{\alpha} = -|x|$$
$$y_{\beta} = y_{\beta} = 0,\qquad u_{(1)}(w(t)) = 0 \qquad (k=1) \tag{2.13}$$

$$x/|x| = x(0)/|x(0)|,\quad |x|^{\cdot} = y_{\alpha} < 0,\quad y_{\alpha} = 0$$
$$y_{\beta} = y_{\beta} = 0,\qquad u_{(1)}(w(t)) = 0 \qquad (k=0) \tag{2.14}$$

The trajectories of these equations realize the times $T[u_{(1)}, 0]$ in accordance with (2.5). This completes the proof of Lemma 2.1.

3. The scheme for the construction of the control $v_0(w)$, used in [4] for the one-dimensional analog of the problem being considered here, can be generalized in a natural way to $n = 3$ and leads to forming in the regions $D_{(1),(0)}$ the impulse controls

$$v_{(1)}(w) = \nu_{1(1)}(w)\,\delta = \nu y_{\alpha}\psi^{(1)}\delta j_{\alpha} + \nu(y_{\beta} + |x|)\,\psi^{(1)}\delta j_{\beta}$$
$$v_{(0)}(w) = \nu_{1(0)}(w)\,\delta = \nu y_{\alpha}\psi^{(0)}\delta j_{\alpha} + \nu y_{\beta}\psi^{(1)}\delta j_{\beta}$$
$$\psi^{(1)} = (y^2 + x^2)^{-1/2},\qquad \psi^{(0)} = |y|^{-1}$$

Theorem 3. The controls $v_{(0),(1)}(w)$ solve Problem 2 in the regions $D_{(1),(0)}$, i.e. the inclusions

$$v_{(1),(0)}(w) \in v_0(w),\qquad D_{(1),(0)}(w) \in W_0(w) \tag{3.1}$$

are valid.

Proof. It is obvious that the collection of possible vectors $w^{(2.1)}(w)$ coincides, when $v = v_{(1),(0)}(w)$ with the collection of possible vectors $w^{(2.1)}(w^{(2)}(w,\ v_{(1),(0)}(w)))$. Therefore, the vector $w^{(2)}(w, v_{(1),(0)}(w))$ can be considered to be the initial position and the first player's possibilities at it can be discussed. We can directly verify the relations

$$R^{(1),(0)}(w) = R^{(1),(0)}(w^{(2)}(w,\ v_{(1),(0)}(w))) = \mu - \sqrt{(y^{(2)})^2 + k^2x^2} \leqslant 0 \quad (k=1,0) \tag{3.2}$$

$$(R^{(1),(0)})^{\cdot} = -|u| - \psi^{(1),(0)}(y_{\alpha}u_{\alpha} + y_{\beta}u_{\beta}) \leqslant 0 \tag{3.3}$$

where $(R^{(1),(0)})^{\cdot}$ are the right derivatives of the functions $R^{(1),(0)}$ by virtue of system (1.1) under a finite control u. For $k = 1$, $|x| \neq 0$ (3.3) becomes a strict inequality, while equality in (3.3)is achieved only in the case $k|x| = 0$ by the control $u = -my$ $(m > 0)$ antiparallel to the velocity. The estimate (3.3) shows that the inclusion $w(t>0,u) \in D_{(1)}(w) \not\ni M$ is preserved for $t > 0$ .The proof is complete for $k = 1$.

For $k = 0$ estimate (3.3) ensures the inclusion $w\ (t > 0,\ u,\ w\ (0)) \in D_{(0,1)} \not\in M$ for $w\ (0) \in D_{(0,1)}\ (w)$. It remains to discuss the cases $w\ (0) \in D_{(0,2)}\ (w)$, $w\ (0) \in D_{(0,3)}\ (w)$. In the first case any control u, holding a position on the surface $R^{(0)}\ (w) = 0$, is antiparallel to vector y and preserves the quantity $y_{\beta}\ |x|\ /\ |y|$ equal to the projection of vector x onto the normal to vector y, lying in the plane containing the vectors x, y. In the second case this projection equals zero, but the admissible control $u = -my$ preserves the inequality $y_{\alpha} \geqslant 0$ and the quantity $|x| > 0$ is not diminished. Thus, an admissible control, preserving the equality $R^{(0)}\ (w) = 0$, cannot lead a trajectory onto M. On the other hand, according to (3.3), any control $u \neq -my\ (m>0)$ leads the trajectory into the region $D_{(0,1)}\ (w)$, whence it is also impossible to go onto M. The proof of Theorem 3 is complete. Let us discuss the possibilities of the second player in the region $D^{(1),(0)}(w)$.

Lemma 3.1 contains three assertions.

3.1.1. Any impulse control $v = \nu_1\delta$ preserves the inclusion $w^{(2)} \in D^{(1),(0)}$

3.1.2. The estimates

$$p^{(1),(0)}(w^{(2)}) \leqslant p^{(1),(0)}(w) \tag{3.4}$$

are valid, becoming equalities only by the controls

$$v^{(1),(0)}(w, n) = -nh_\alpha^{(1),(0)}(w)\,\delta j_\alpha - nh_\beta^{(1),(0)}(w)\,\delta i,$$

$$0 \leqslant n \leqslant \nu, \quad w \in C^{(1),(0)}\,[(l^{(1),(0)})^2 = y_\beta^2 + (\lambda^{(1),(0)}(w))^2 > 0]$$

$$h_\alpha^{(1),(0)} = \lambda^{(1),(0)}/l^{(1),(0)}, \quad h_\beta^{(1),(0)} = -y_\beta/l^{(1),(0)}, \quad l^{(1),(0)} > 0$$

$$v^{(1),(0)}(w, n) = n\delta j_\alpha, \quad 0 \leqslant n \leqslant \nu, \quad w \in E^{(1),(0)}\,[l^{(1),(0)} = 0]$$

3.1.3. The equalities

$$h_\alpha^{(1),(0)}(w) = h_\alpha^{(1),(0)}(w^{(2)}), \quad h_\beta^{(1),(0)}(w) = h_\beta^{(1),(0)}(w^{(2)}) \quad \text{for } w \in C^{(1)(0)}$$

$$y_\beta = y_\beta^{(2)} = 0 \quad \text{for } w \in E^{(1),(0)} \tag{3.5}$$

are valid for $v = v^{(1),(0)}$

Proof. We carry it out only for the case $k = 1$, $w \in C^{(1),(0)}$ The remaining cases can be considered analogously. By $\varphi_*^{[p]}(w, p^{(1),(0)}) \geqslant 0$ we denote the partial derivative of the function $\varphi_*(w, p)$ obtained after the substitution $p = y_\alpha + \lambda$ into the function $\varphi(w, \lambda)$; note that $\varphi_*^{[p]} > 0$ is strictly positive for $R^{(1)}(w) > 0$ We restrict ourselves only to the subcase $R^{(1)}(w) > 0$ and we start to vary $v = \nu_1\delta$ from zero. Then the root variation $\delta p^{(1)} = p^{(1)}(w^{(2)}) - p^{(1)}(w)$ changes, for small $|\nu_1|$, in accordance with the equation

$$\varphi_*^{[p]}(w, p^{(1)})\,\delta p^{(1)} = -|\nu_1| - h_\alpha^{(1)}(w)\,\nu_\alpha - h_\beta^{(1)}(w)\,\nu_\beta + O(w, |\nu_1|) \leqslant 0 \tag{3.6}$$

The sum of the first-dimension terms are strictly negative for $\nu_1\delta \neq v^{(1)}(n, w)$. This sum vanishes only for $v = v^{(1)}(n, w)$, but here it is easily verified that the quantity $O(w, |\nu_1|)$ also vanishes. Let the vector ν_1 be finite and satisfy the relations

$$\nu_1 \neq v^{(1),(0)}, \qquad w^{(2)}(w, \theta, \nu_1) \in C^{(1),(0)} \quad \text{for} \quad 0 \leqslant \theta \leqslant 1$$

Let us assume to the contrary that $\delta p^{(1)}(w, \nu_1) \geqslant 0$. From the relation $\nu_1 \neq v^{(1),(0)}$ and from estimate (3.6) follows the bound $\delta p^{(1)}(w, \theta\nu_1) < 0$ for sufficiently small θ. The latter bound, together with the contrary assumption, implies the existence of a number $0 < \theta_1 < 1$ such that $\delta p^{(1)}(w^{(2)}(w, \theta_1\nu_1), \Delta\theta\nu_1) > 0$ for any sufficiently small $\Delta\theta > 0$. This inequality contradicts estimate (3.6) applied at the point $w^{(2)}(w, \theta_1\nu_1)$. Thus, we have proven estimate (3.4) and the first part of assertion 3.1.2. For $R^{(1)} > 0$ the inclusion $w^{(2)} \in D^{(1)}$ is a consequence of the presence of the root $\lambda^{(1)}(w^{(2)}) = p^{(1)}(w^{(2)}) - y_\alpha^{(2)}$ This proves assertion 3.1.1. Equality (3.5) is verified by a substitution. The proof of Lemma 3.1 is complete.

Lemma 3.2 contains two assertions.

3.2.1. Any impulse control $u = \mu_1\delta$, preserving the inclusion $w^{(2.1)} \in D^{(1),(0)}$, cannot violate the estimates

$$p^{(1),(0)}(w^{(2.1)}) \geqslant p^{(1),(0)}(w^{(2)}) \tag{3.7}$$

which become strict equalities only for

$$u^{\{(1),(0)\}}(w^{(2)}, r) = ru_{(1)}(w^{(2)}, k=1, 0), \qquad 0 \leqslant r \leqslant 1$$

3.2.2. The controls $u^{\{(1),(0)\}}(w^{(2)}, r)$ preserve the quantities

$$h_\alpha^{(1),(0)}(w^{(2)}) = h_\alpha^{(1),(0)}(w^{(2.1)}(w^{(2)}, u^{\{(1),(0)\}}(w^{(2)}, r)))$$

$$h_\beta^{(1),(0)}(w^{(2)}) = h_\beta^{(1),(0)}(w^{(2.1)}(w^{(2)}, u^{\{(1),(0)\}}(w^{(2)}, r)))$$

for $w^{(2)} \in C^{(1),(0)}(w^{(2)})$.

The proof is analogous to that of Lemma 3.1.

In summary, Lemmas 3.1 and 3.2 establish the equality

$$\max_v \min_{u(v)} [p^{(1),(0)}(w^{(2,1)}) - p^{(1),(0)}(w)] = 0 \tag{3.8}$$

where the minimum is sought on all $u(w)$ which are impulsive and which preserve the inclusion $w^{(2,1)} \in D^{(1),(0)}$, while the maximum, on all admissible controls

$$v = \nu_1\delta, \quad \nu - |\nu_1| \geqslant 0.$$

4. Thus, the control

$$u_{(1)}(w^{(2)}) = u^{(1),(0)}(w^{(2)}) = \lambda^{(1),(0)}(w^{(2)})\,\delta j_\alpha - y_\beta^{(2)}\delta j_\beta, \qquad w^{(2)} \in C^{(1),(0)} \tag{4.1}$$

has been formed in the region $C^{(1),(0)}$. If however, at the position $w \in E^{(1),(0)}$ the second player uses a finite control $v(w)$, then $w^{(2)} = w$, and a natural generalization of the impulse control $u_{(1)}(w^{(2)})$ in this case is the control

$$u^{(1),(0)}(w, v) = u_\alpha^{(1)(0)}(w, v)\, j_\alpha - v_\beta j_\beta - v_\gamma j_\gamma \tag{4.2}$$

where $u_\alpha^{(1),(0)}(w, v)$ are the smallest roots of the equation

$$|v| - \sqrt{v_\beta^2 + v_\gamma^2 + u_\alpha^2} - y_\alpha(v_\alpha + u_\alpha)\xi^{-1} = 0 \tag{4.3}$$

Equation (4.3) admits of the obvious roots $u_{\alpha 1,1} = u_{\alpha 0,1} = -v_\alpha$ and, furthermore, of the root $u_{\alpha 1,2} = 2(v_\alpha\xi^2 - |v|\, y_\alpha\xi)\, q^{-1} - v_\alpha$ when $|v|\, q - 2y_\alpha\xi v_\alpha - 2|v|\, y_\alpha^2 \geqslant 0$ and the root $u_{\alpha 1,2} = 0$ when $v_\beta = v_\gamma = 0$, $v_\alpha < 0$.

5. Let us begin the analysis of the case $k = 1$ and, unless we stipulate otherwise, we shall consider only this case. We introduce the notation

$$z^{(1)}(w) = p^{(1)}(w)\,|x|^{-1}, \qquad \zeta = \xi\,|x|^{-1} \tag{5.1}$$

$$z_\alpha = y_\alpha\,|x|^{-1}, \qquad z_\beta = y_\beta\,|x|^{-1}$$

$$z^{(1)(2)}(w^{(2)}) = p^{(1)}(w^{(2)})\,|x|^{-1}, \qquad \zeta^{(2)} = \xi^{(2)}\,|x|^{-1}$$

$$z_\alpha^{(2)} = y_\alpha^{(2)}\,|x|^{-1}, \qquad z_\beta^{(2)} = y_\beta^{(2)}\,|x|^{-1}$$

$$0 \leqslant b^{(1)}(w^{(2)}) = \{(z_\beta^{(2)})^2 - (z^{(1)(2)}(w^{(2)}) - z_\alpha^{(2)})^2\}^{1/2}$$

The function $T^{(1)}(z^{(1)}(w))$ increases strictly monotonically with respect to the argument $z^{(1)}(w)$. However, according to Lemma 3.1 (Lemma 3.2) the second (first) player cannot increase (decrease) the quantity $z^{(1)}(w)$ and the function $T^{(1)}(z^{(1)}(w))$ by means of impulse controls. As a corollary of this it is natural to examine the consequences of impulse actions of the second and first players who at the initial instant have applied the impulse controls

$$v^{(1)}(w, 0 \leqslant n \leqslant \nu), \qquad u^{\{(1)\}}(w^{(2)}, 0 \leqslant r \leqslant 1) = ru_{(1)} \tag{5.2}$$

and who for $t > 0$ apply the finite controls u, v. It is not difficult to verify the validity of the equalities

$$\zeta^{(2.1)} = \zeta - (b^{(1)} - b^{(2)}) = \zeta - a, \qquad z_\alpha^{(2,1)} = z_\alpha + ah_\alpha(w) \tag{5.3}$$

$$z_\beta^{(2,1)} = z_\beta + ah_\alpha(w), \qquad 0 \leqslant b^{(2)} = n / |x| \leqslant \nu / |x| \tag{5.4}$$

$$z^{(1)}(w) = z^{(1)}(w^{(2,1)})$$

$$h_\alpha(w) = h_\alpha(w^{(2,1)}), \qquad h_\beta(w) = h_\beta(w^{(2,1)}) \tag{5.5}$$

In formulas (5.3) we can consider the case

$$w \in C^{(1)}[y_\beta^2 + (\lambda^{(1)}(w))^2 > 0] \cap [R^{(1)}(w) > 0] \tag{5.6}$$

From (2.2), (5.1) - (5.4) it follows that the function $z^{(1)}(w) = z^{(1),(2,1)}(w^{(2,1)})$ is the smallest root of the equation

$$\zeta^{(2,1)} - \sqrt{(z_\beta^{(2,1)})^2 + ((z - z_\alpha)^{(2,1)})^2} - \sqrt{z^2 + 1} = 0 \tag{5.7}$$

As a consequence of system (4.1), for the variables ζ, z_α, z_β we can obtain the differential equations

$$\zeta^{\cdot} = -\zeta z_\alpha - |u_1| + |v_1|, \qquad z_\alpha^{\cdot} = -1 - z_\alpha^2 + z_\beta^2 + u_{1\alpha} + v_{1\alpha} \tag{5.8}$$

$$z_\beta = -2z_\alpha z_\beta + u_{1\beta} + v_{1\beta}, \qquad u_1 = u|x|^{-1}, \qquad v_1 = v|x|^{-1}$$

The right derivative of the function $T^{(1)}(z^{(1)}(w^{(2,1)}))$ along the motion starting from the position $w^{(2.1)}$ has the form

$$[T^{(1)}(z^{(1)}(w^{(2,1)}))]^{\cdot} = (dT^{(1)} / dz^1)[z^{(1)}(w^{(2.1)})]^{\cdot}$$

$$[z^1(w^{(2,1)})]^{\cdot} = P_1(w^{(2.1)}) + P_2(w^{(2,1)}, u) + P_3(w^{(2,1)}, v)$$

where $P_1(w^{(2,1)})$ is a combination of terms not depending on the controls, $P_2(w^{(2,1)}, u)$ of terms depending only on u, while $P_3(w^{(2,1)}, v)$ of terms depending only on v. To compute the indicated quantities we differentiate Eq. (5.5) with respect to time, and we obtain the equation

$$(z^{(1)}(w^{(2,1)}))^{\cdot}\,\theta(w) = (-\zeta^{(2.1)})^{\cdot} - h_\alpha(w)(z_\alpha^{(2,1)})^{\cdot} - h_\beta(w)(z_\beta^{(2,1)})^{\cdot} \tag{5.9}$$

$$\theta(w) = [-h_\alpha(w) - z^{(1)}(w) / \sqrt{(z^{(1)}(w))^2 + 1}] > 0 \tag{5.10}$$

The estimate (5.10) is a consequence of the assumption $R^{(1)}(w) > 0$. The equality

$$P_1 = (\theta(w))^{-1}[H(w) + (\zeta h_\alpha + z_\alpha)a] \tag{5.11}$$

$$H(w) = \zeta z_\alpha - h_\alpha(z_\beta^2 - z_\alpha^2 - 1) - 2h_\beta z_\alpha z_\beta$$

is a corollary of (5.9), (5.10) and (5.6). Simple calculations allow us to establish the estimate

$$\zeta h_\alpha + z_\alpha \leqslant 0 \tag{5.12}$$

while arguments analogous to those in the proofs of Lemmas 3.1, 3.2 establish the equality

$$\max_v \min_{u(v)} [P_2(w^{(2,1)}, u) + P_3(w^{(2,1)}, v)] = 0 \tag{5.13}$$

From (5.12) follows the estimate

$$P_1(w, u_{(1)}(w^{(2)}), \quad v^{(1)}(0 \leqslant n \leqslant \nu, w)) \leqslant P_1(w^{(2,1)}) \tag{5.14}$$

$$w^{(2,1)} = w^{(2,1)}(w, u^{(1)}, v^{(1)})$$

6. Theorem 6. The equalities

$$T[u^\circ, v^\circ] = T\,[[u_{(1)}(w^{(2)}),\ u^{(1),(0)}(w, v)],\ v^{(1),(0)}(w, n)] = T^{(1),(0)}(z^{(1),(0)}(w)) \quad (6.1)$$

$$[u_{(1)}(w^{(2)}),\ u^{(1),(0)}(w, v)] = u^\circ(w, v) \quad (6.2)$$

$$v^{(1),(0)}(w, n) = v^\circ(w)$$

$$M \cup D^{(1),(0)}(w) = W^\circ(w), \qquad D_{(1),(0)}(w) = W_0(w) \quad (6.3)$$

follow from the inclusions $w \in D^{(1),(0)}(w)$. From (6.1) it follows that the controls $[u_{(1)}, u^{(1),(0)}]$, $v^{(1),(0)}$ solve Problem 1 in the regions $D^{(1),(0)}(w)$ with optimal times $T^{(1),(0)}(z^{(1),(0)}(w))$, while in the remaining part of the position space the second player can solve Problem 2 (of escape).

Proof. Once again we restrict ourselves to the case $k = 1$. Because the function $T^{(1)}(z^{(1)}(w))$ is monotonic with respect to $z^{(1)}(w)$ and from Lemmas 3.1, 3.2, there follows the equality

$$\max_{\nu_1} \min_{\mu(\nu_1)} \Delta T^{(1)} = \Delta T(w, u^{(1)}, v^{(1)}) = 0 \quad (6.4)$$

From (5.13) and (5.14) follows the equality

$$\max_v \min_{u(v)} (T^{(1)})^\cdot = T^{(1)\cdot}(w, u_{(1)}, v^{(1)}) = -1 \quad (6.5)$$

Equalities (6.4), (6.5) prove the equalities (6.1), (6.2) and the inclusion $D^{(1),(0)} \in \in W^\circ(w)$. Equalities (6.3) follow from the latter inclusion, from inclusion (3.4) and from the equality $M \cup D^{(1),(0)} \cup D_{(1),(0)} = W$. The proof of Theorem 6 is complete. The case $k = 0$ can be treated analogously.

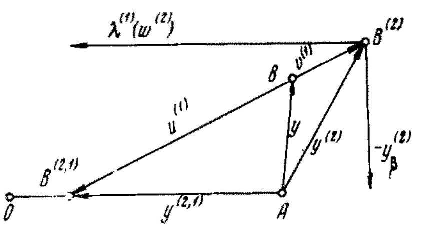

Fig. 1.

The optimal motion is shown in the Fig. 1. The vectors $(O, A) = R$ and $(O, B) = y$ represent the initial position and velocity. The vector $v^{(1)} = (B, B^{(2)})$ in sum with the vector y forms the vector $y^{(2)} = (A, B^{(2)})$ — the result of the second player's optimal actions. The vector $(B^{(2)}, B^{(2,1)}) = u^{(1)}$ in sum with the vector $y^{(2)}$ leads to the vector $y^{(2,1)}$ directed along the vector (O, A). After this the optimal motion starts from the position $w^{(2,1)}$ and takes place along the fixed straight line (O, A) during a time $T^{(1)}$.

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REDUCTION OF THE DIFFERENTIAL EQUATIONS OF NONHOLONOMIC MECHANICS TO LAGRANGE FORM

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We consider the problem of the equivalence of a certain system of ordinary differential equations to a system of Lagrange equations. Wherever we do not expressly say so, we have in mind sationary nonholonomic Chaplygin systems with linear constraints. The equations of motion of non-holonomic systems in the Routh form, Chaplygin in appearance, differ from the Lagrange equations of the second kind in the presence of additional terms (constraint reactions, nonholonomic terms). This fact hinders the extension of integration methods of equations of motion of holonomic systems to nonholonomic ones. The few attempts [1, 2] to seek general methods for integrating the equations of nonholonomic mechanics were reduced to the transformation of the equations of motion to Lagrange form [3]. The equations of motion of nonholonomic systems have the form of Lagrange equations [1, 4] only in exceptional cases.

The problem of determining the conditions which guarantee the equivalence of a given system of differential equations

$$F_j(q_1^{\cdot\cdot},\ldots,q_n^{\cdot\cdot},q_1^{\cdot},\ldots,q_n^{\cdot},q_1,\ldots,q_n,t)=0 \qquad (j=1,\ldots,n) \tag{0.1}$$

with the Lagrange system

$$L_j(\theta)=0 \qquad (j=1,\ldots,n) \qquad \left(L_j=\frac{d}{dt}\frac{\partial}{\partial q_j^{\cdot}}-\frac{\partial}{\partial q_j}\right) \tag{0.2}$$

where θ is a certain function of $q_1^{\cdot}, \ldots, q_n^{\cdot}, q_1, \ldots, q_n, t$, is a familiar one. Necessary and sufficient conditions (the Helmholtz conditions) were obtained in [5 - 7] on whose basis we can determine from the appearance of Eqs. (0.1) whether each of these equations individually is a Lagrange equation relative to the function θ called the Helmholtz kinetic potential. It should be noted that the Helmholtz conditions applied by Chaplygin are usually not fulfilled for Routh equations [8, 9], nevertheless, in some cases, by combining these equations they can be replaced by an equivalent Lagrange system [8]. A theorem was proven in [10] that the equations of motion of a mechanical system with linear nonintegrable constraints

$$w_i=q_i^{\cdot}+\sum_{s=k+1}^{n}a_{is}(q_1,\ldots,q_n,t)\,q_s^{\cdot}+a_i(q_1,\ldots,q_n,t)=0 \quad (i=1,\ldots,k<n)$$